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(I)-ENVELOPES OF CLOSED CONVEX SETS IN BANACH SPACES

BY

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ABSTRACT

We study the notion of (I)-generating introduced by V. Fonf and J. Lindendstrauss and a related notion of (I)-envelope. As a consequence of our results we get an easy proof of the James characterization of weak compactness in Banach spaces with weak* angelic dual unit ball and an easy proof of the James characterization of reflexivity within a large class of spaces. We also show by an example that the general James theorem cannot be proved by this method.

1. Introduction

Let K be a weak* compact convex subset of a dual Banach space X^* and $B \subset K$. Then the set B **(I)-generates** K if for every representation of B as $\bigcup_{n=1}^{\infty} C_n$ we have that K is the norm closed convex hull of $\bigcup_{n=1}^{\infty} \overline{\operatorname{conv} C_n}^{w^*}$.

This notion was introduced by V. Fonf and J. Lindenstrauss in [5]. The main application is their Theorem 2.3 which says that B (I)-generates K whenever B is a boundary of K (i.e., whenever the function $k \mapsto k(x)$ attains its maximum on K at some point of B, for each $x \in X$). As a consequence of this theorem one gets a separable version of the James theorem, even in a stronger form (see [5]).

In the present paper, we study the notion of (I)-generation in nonseparable Banach spaces. Let us start by introducing the following natural concept of (I)-envelope.

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Let X be a Banach space and $B \subset X^*$, the **(I)-envelope** of B is defined by the formula

$$\text{(I)-env}(B) = \bigcap \left\{ \overline{\operatorname{conv} \bigcup_{n=1}^{\infty} \overline{\operatorname{conv} C_n}^{w^*}}^{\|\cdot\|} : B = \bigcup_{n=1}^{\infty} C_n \right\}. \tag{1}$$

Clearly, an equivalent description is given by

$$\text{(I)-env}(B) = \bigcap \left\{ \overline{\bigcup_{n=1}^{\infty} \overline{\operatorname{conv} C_n}^{w^*}}^{\|\cdot\|} : C_n \nearrow B \right\}. \tag{2}$$

Remarks 1.1:

(i) The (I)-envelope of B is a norm closed convex set and we obviously have

$$\overline{\operatorname{conv} B}^{\|\cdot\|} \subset \text{(I)-env}(B) \subset \overline{\operatorname{conv} B}^{w^*}.$$

(ii) If B is norm separable, then it is easy to check that

$$\text{(I)-env}(B) = \overline{\operatorname{conv} B}^{\|\cdot\|}$$

(cf., the proof of [5, Proposition 2.2(a)] or the proof of the separable version of the James theorem below).

(iii) One can easily check that for any B we have

$$\text{(I)-env}(\text{(I)-env}(B)) = \text{(I)-env}(B),$$

and hence one can also study **(I)-convex sets**, i.e., those sets B for which the (I)-envelope of B is just B. Then the (I)-envelope of B is the smallest (I)-convex set containing B.

(iv) In the (1) and (2) it is enough to consider only bounded sets C_n. Indeed, in (1) we can replace the sequence $(C_n)_{n\in\mathbb{N}}$ by the sequence $(C_n \cap mB_{X^*})_{m,n\in\mathbb{N}}$ and in (2) we can replace the sequence $(C_n)_{n\in\mathbb{N}}$ by the sequence $(C_n \cap nB_{X^*})_{n\in\mathbb{N}}$.

The above mentioned result of [5] can be now expressed as follows:

Let X be a Banach space, $K \subset X^*$ a weak compact convex set and $B \subset K$ a boundary of K. Then $K = \text{(I)-env}\, B$.*

Let us recall the proof of the separable version of the James theorem using this result (cf., [6, Theorem 5.7]):

Let X be a separable Banach space and $B \subset X$ a bounded closed convex set such that each $f \in X^*$ attains its maximum on B. Then B is a boundary of $K = \overline{B}^{w^*}$ (consider X canonically embedded into X^{**}), hence $K = \text{(I)-env}(B)$.

However, it easily follows from the separability of X that the (I)-envelope of B is just B: Let $\{c_n\colon n \in \mathbb{N}\}$ be a countable dense subset of B. Then, for each $\varepsilon > 0$, we have

$$\text{(I)-env}(B) \subset \overline{\operatorname{conv} \bigcup_{n\in\mathbb{N}} c_n + \varepsilon B_{X^{**}}}^{\|\cdot\|} \subset B + \varepsilon B_{X^{**}}.$$

As $\varepsilon > 0$ is arbitrary, (I)-env$(B) = B$. Therefore $B = \overline{B}^{w^*}$, hence B is weakly compact.

By a slight modification of the above argument one gets that (I)-env$(B) = B$ for any closed convex bounded subset B of a subspace of a weakly compactly generated space (instead of separability one uses the characterization of subspaces of WCG given in [4]). This yields an easy proof of the James theorem within subspaces of WCG. One is tempted to try to get by this method an easy proof of the general James theorem. This inspires the following question.

Question 1.2: *Let X be a Banach space, $B \subset X$ be a bounded closed convex set such that* (I)-env(B) $= \overline{B}^{w^*}$ *(X is considered canonically embedded into X^{**}). Is B then weakly compact?*

The positive answer to our question would yield an alternative proof of the James theorem. Unfortunately (or, perhaps, fortunately), one of the main results of this paper (Example 4.1 below) shows that the answer is in general negative. We also get some positive results — we show that the answer is positive in many cases.

In Sections 2–4 we consider only real Banach spaces without mentioning it explicitly. However, the situation in complex Banach spaces is completely analogous as we will see in Section 5.

2. The key lemma and its consequences

A key tool for proving our results is given in the following characterization of the (I)-envelope.

Lemma 2.1: *Let Y be a Banach space, $B \subset Y^*$ and $\eta \in \overline{\operatorname{conv} B}^{w^*}$. Then the following assertions are equivalent.*

(1) $\eta \notin$ (I)-env(B).

(2) *There is a sequence of $x_n \in B_Y$ such that*

$$\sup_{\xi\in B} \limsup_{n\to\infty} \xi(x_n) < \inf_{n\in\mathbb{N}} \eta(x_n).$$

(3) *There is a sequence of $x_n \in B_Y$ such that*

$$\sup_{\xi\in B}\limsup_{n\to\infty}\xi(x_n) < \liminf_{n\to\infty}\eta(x_n).$$

(4) *There is a sequence of $x_n \in B_Y$ such that*

$$\sup_{\xi\in B}\limsup_{n\to\infty}\xi(x_n) < \limsup_{n\to\infty}\eta(x_n).$$

Proof: First note that the implications 2⇒3⇒4 are trivial. The implication 4⇒2 can be easily proved by choosing a suitable subsequence of x_n's. Hence the assertions (2), (3) and (4) are equivalent.

Let us show that 2⇒1. Let $x_n \in B_Y$ satisfy the inequality in the assertion (2). Choose real numbers c and d such that

$$\sup_{\xi\in B}\limsup_{n\to\infty}\xi(x_n) < c < d < \inf_{n\in\mathbb{N}}\eta(x_n).$$

Set

$$C_n = \{\xi \in B\colon (\forall k \geq n)(\xi(x_k) \leq c)\}, \quad n \in \mathbb{N}.$$

Then each C_n is a closed subset of B and $C_n \nearrow B$. For each $n \in \mathbb{N}$ we have

$$\overline{\operatorname{conv} C_n}^{w^*} \subset \{\xi \in \overline{\operatorname{conv} B}^{w^*}\colon (\forall k \geq n)(\xi(x_k) \leq c)\}.$$

If $\xi \in \bigcup_{n\in\mathbb{N}} \overline{\operatorname{conv} C_n}^{w^*}$ we get

$$\|\eta - \xi\| \geq |\eta(x_n) - \xi(x_n)| \geq d - c,$$

provided n is large enough. It follows that $\eta \notin$ (I)-env(B), hence the assertion (1) is fulfilled.

Finally we show that 1⇒3. Suppose that $\eta \in \overline{\operatorname{conv} B}^{w^*} \setminus$ (I)-env(B). Then there are C_n, $n \in \mathbb{N}$, bounded subsets of B such that $C_n \nearrow B$, and $c > 0$ such that $\operatorname{dist}(\eta, \overline{\operatorname{conv} C_n}^{w^*}) > c$ for each $n \in \mathbb{N}$. To finish the proof we need the following claim.

Claim: *Let Y be a Banach space and $K \subset Y^*$ be a nonempty convex weak* compact set. Then*

$$\operatorname{dist}(0, K) = \sup_{y\in B_Y}\inf_{k\in K} k(y).$$

Since this equality can be expressed as

$$\inf_{k\in K}\sup_{y\in B_Y} k(y) = \sup_{y\in B_Y}\inf_{k\in K} k(y),$$

the claim follows immediately from [14, Remark 6 and Lemma 11].

If we apply the claim to $K = \eta - \overline{\operatorname{conv} C_n}^{w^*}$ (note that K is weak* compact as C_n is bounded), we get that there exists $x_n \in B_Y$ such that

$$\eta(x_n) - \sup_{\xi \in C_n} \xi(x_n) > c.$$

By passing to a subsequence we may suppose that the sequence $\eta(x_n)$ converges. Then we have

$$c \leq \liminf_{n\to\infty}(\eta(x_n) - \sup_{\xi\in C_n} \xi(x_n)) = \lim_{n\to\infty} \eta(x_n) - \limsup_{n\to\infty} \sup_{\xi\in C_n} \xi(x_n),$$

hence

$$\limsup_{n\to\infty} \sup_{\xi\in C_n} \xi(x_n) \leq \lim_{n\to\infty} \eta(x_n) - c.$$

If $\xi_0 \in B$ is arbitrary, then $\xi_0 \in C_n$ for n large enough. Thus

$$\limsup_{n\to\infty} \xi_0(x_n) \leq \limsup_{n\to\infty} \sup_{\xi\in C_n} \xi(x_n),$$

and so

$$\sup_{\xi\in B} \limsup_{n\to\infty} \xi(x_n) \leq \lim_{n\to\infty} \eta(x_n) - c,$$

which shows that the assertion (3) is satisfied. ∎

Remark 2.2: If $K \subset Y^*$ is a convex weak* compact set and $B \subset K$ is a boundary of K, then by a consequence of the Simons inequality (see [13, Theorem 3]) we get

$$\sup_{\xi\in B} \limsup_{n\to\infty} \xi(x_n) \geq \limsup_{n\to\infty} \eta(x_n),$$

for any $\eta \in K$ and any bounded sequence of $x_n \in Y$. Therefore, by Lemma 2.1 we have $K = \text{(I)-env}(B)$. This gives an alternative proof of [5, Theorem 2.3].

We will study mainly the (I)-envelopes of closed convex subsets of a Banach space considered as subsets of the second dual. Therefore, we give the formulation of Lemma 2.1 for this special case. It is an immediate consequence of Lemma 2.1.

LEMMA 2.3: *Let X be a Banach space, $B \subset X$ a closed convex set and $G \in \overline{B}^{w^*}$. Then the following assertions are equivalent.*

(1) $G \notin \text{(I)-env}(B)$;
(2) *there is a sequence of $\xi_n \in B_{X^*}$ such that*

$$\sup_{x\in B} \limsup_{n\to\infty} \xi_n(x) < \inf_{n\in\mathbb{N}} G(\xi_n);$$

(3) *there is a sequence of* $\xi_n \in B_{X^*}$ *such that*

$$\sup_{x\in B}\limsup_{n\to\infty}\xi_n(x) < \liminf_{n\to\infty} G(\xi_n);$$

(4) *there is a sequence of* $\xi_n \in B_{X^*}$ *such that*

$$\sup_{x\in B}\limsup_{n\to\infty}\xi_n(x) < \limsup_{n\to\infty} G(\xi_n).$$

As a consequence of this lemma we get the following.

PROPOSITION 2.4: *Let* X *be a Banach space and* $B \subset X$ *be a closed convex set.*

(i) *Any element of* (I)-env(B) *is weak* sequentially continuous.*

(ii) *If* B_{X^*} *is weak* sequentially compact, then any weak* sequentially continuous element of* $\overline{B}^{w^*}$ *belongs to* (I)-env(B).

Proof: (i) Suppose $G \in \overline{B}^{w^*} \setminus B$ is not weak* sequentially continuous. Then there is $\eta \in X^*$ and a sequence $\eta_n \in X^*$ weak* converging to η such that $G(\eta_n) \not\to G(\eta)$. By the uniform boundedness principle we can assume that all η_n's belong to B_{X^*}. Set $\xi_n = \frac{1}{2}(\eta_n - \eta)$. Then $\xi_n \in B_{X^*}$, the sequence ξ_n weak* converges to 0 and $G(\xi_n) \not\to 0$. Without loss of generality we can assume that $\inf_{n\in\mathbb{N}} G(\xi_n) > 0$. Then, by Lemma 2.3(2⇒1), $G \notin$ (I)-env(B).

(ii) Suppose that B_{X^*} is weak* sequentially compact and

$$G \in \overline{B}^{w^*} \setminus \text{(I)-env}(B)$$

is weak* sequentially continuous. Then there is, by Lemma 2.3, a sequence of $\xi_n \in B_{X^*}$ such that

$$\sup_{x\in B}\limsup_{n\to\infty}\xi_n(x) < \inf_{n\in\mathbb{N}} G(\xi_n).$$

As B_{X^*} is weak* sequentially compact, there is a subsequence of ξ_n which weak* converges. As the above inequality is preserved by passing to a subsequence (note that the left-hand side cannot increase and the right-hand side cannot decrease), we can, without loss of generality, suppose that ξ_n weak* converges to some $\xi \in B_{X^*}$. Then

$$\sup_{x\in B}\limsup_{n\to\infty}\xi_n(x) = \sup_{x\in B}\xi(x)$$

and, as G is weak* sequentially continuous,

$$\inf_{n\in\mathbb{N}} G(\xi_n) \le \lim_{n\to\infty} G(\xi_n) = G(\xi).$$

Finally, $G \in \overline{B}^{w^*}$, and hence

$$G(\xi) \leq \sup_{x \in B} \xi(x),$$

which is a contradiction completing the proof. ∎

The assumption that B_{X^*} is weak* sequentially compact in the assertion (ii) of the previous proposition cannot be omitted. It is witnessed by the following example.

Example 2.5: Let $X = \ell_\infty$ and B be the unit ball of c_0 (note that c_0 is a subspace of ℓ_∞). Then (I)-env$(B) = B$, B is not weakly compact and any element of X^{**} is weak* sequentially continuous.

Proof: Since B is separable, we have (I)-env$(B) = B$ (cf., Remark 1.1). Further, B is not weakly compact as c_0 is not reflexive. Finally, the space ℓ_∞ has the Grothendieck property, i.e. weak and weak* convergence of sequences coincide in X^* (see, [2, Theorem VII.15]). Therefore, any element of X^{**} is weak* sequentially continuous. ∎

3. (I)-envelopes of closed convex sets

In this section we collect partial positive answers to Question 1.2. We consider any Banach space canonically embedded into its second dual and the (I)-envelopes of closed convex sets are done in the second dual.

The first theorem contains a positive answer to Question 1.2 within the Banach spaces with angelic dual unit ball. Recall that a compact space K is **angelic** if closures in K can be described by converging sequences (i.e., if whenever $A \subset K$ and $x \in \overline{A}$, then there is a sequence $x_n \in A$ with $x_n \to x$). A wide class of Banach spaces with angelic dual unit ball is that of weakly Lindelöf determined spaces. A space X is **weakly Lindelöf determined** if there is $M \subset X$ with span M dense in X such that $\{m \in M : \xi(m) \neq 0\}$ is countable for each $\xi \in X^*$. Some properties of this class can be found in [1], [3, Chapter 7] or [8]. The fact that the dual unit balls of spaces from this class are angelic is well-known, a proof can be found in [8, Lemma 1.6 and Theorem 4.17].

THEOREM 3.1: *Let X be a Banach space such that (B_{X^*}, w^*) is angelic. Then (I)-env$(B) = B$ for each closed convex set $B \subset X$. In particular, it is true if X is weakly Lindelöf determined.*

Proof: Suppose $G \in \overline{B}^{w^*} \setminus B$. Then $G \in X^{**} \setminus X$ and hence $G_{|B_{X^*}}$ is not weak* continuous by [7, Corollary 224]. As B_{X^*} is angelic, it follows that G is not weak* sequentially continuous. Hence $G \notin$ (I)-env(B) by Proposition 2.4(i). ∎

The previous theorem contains a stronger result than just a positive answer to Question 1.2. It asserts that the (I)-envelope of any bounded closed convex set B in a space from the specified class is just the set B with no point added. The following example shows that it is not always the case.

Example 3.2: Let X be the space of all continuous functions on the ordinal interval $[0, \omega_1]$ equipped with the maximum metric and B be the closed unit ball of X. Then $B \subsetneqq$ (I)-env$(B) \subsetneqq \overline{B}^{w^*}$.

Proof: Let $X = C([0, \omega_1])$. Then X^* can be canonically identified with $\ell_1([0, \omega_1])$ and X^{**} with $\ell_\infty([0, \omega_1])$. The bidual unit ball $B_{X^{**}}$ is then represented by $[-1, 1]^{[0, \omega_1]}$ and the weak* topology on this ball coincides with the product topology (i.e., pointwise convergence topology). We claim that $\chi_{\{\omega_1\}} \in$ (I)-env$(B_X) \setminus B_X$.

Indeed, let $B_X = \bigcup_{n \in \mathbb{N}} C_n$. Then, for some $n \in \mathbb{N}$, C_n contains $\chi_{(\alpha, \omega_1]}$, for uncountably many $\alpha < \omega_1$. The element $\chi_{\{\omega_1\}}$ is then in the weak* closure of that C_n. It follows that $\chi_{\{\omega_1\}} \in$ (I)-env(B_X). The fact that $\chi_{\{\omega_1\}} \notin B_X$ is obvious.

On the other hand, (I)-env(B_X) is not equal to $B_{X^{**}}$ since

$$\chi_{\{\omega\}} \in B_{X^{**}} \setminus \text{(I)-env}(B_X).$$

The latter fact follows from Proposition 2.4(i) as $\chi_{\{\omega\}}$ is not weak* sequentially continuous. Indeed, the sequence of Dirac measures δ_n, $n < \omega$, weak* converges to δ_ω, while $\chi_{\{\omega\}}(\delta_n) = 0 \not\to 1 = \chi_{\{\omega\}}(\delta_\omega)$. ∎

Remark 3.3: Note that we could show that $\chi_{\{\omega_1\}} \in$ (I)-env(B_X) by referring to Proposition 2.4(ii). Indeed, $\chi_{\{\omega_1\}}$ is weak* sequentially continuous and B_{X^*} is weak* sequentially compact as X is an Asplund space. But the above proof is more direct and elementary.

The following theorem deals with a particular case of Question 1.2; the case when the set B in question is the closed unit ball of X.

THEOREM 3.4: *Let X be a Banach space such that* (I)-env$(B_X) = B_{X^{**}}$. *Then X has the Grothendieck property, i.e., the weak and weak* convergences of sequences coincide in X^*.*

Proof: Suppose that X does not have Grothendieck property. Then there is a sequence $\xi_n \in X^*$ weak* converging to 0 such that $G(\xi_n) \not\to 0$ for some

$G \in B_{X^{**}}$. Then G is not weak* sequentially continuous and hence $G \notin$ (I)-env(B_X) by Proposition 2.4(i). ∎

Reflexive spaces, of course, have the Grothendieck property. However, there are also nonreflexive spaces with this property. By a result of Grothendieck (see [2, Theorem VII.15]) the space ℓ_∞ has the Grothendieck property. Further spaces enjoying the Grothendieck property are, for example, the spaces $\ell_\infty(E)$ of E-valued bounded sequences, where E is a B-convex Banach lattice [10] and weak L_p spaces for $1 < p < \infty$ [11]. However, there are large classes of Banach spaces which contain no nonreflexive space with the Grothendieck property. We collect some significant ones in the following corollaries. We start by a rather technical but quite general one.

Corollary 3.5: *Let X be a Banach space such that* (I)-env$(B_X) = B_{X^{**}}$. *If there is $M \subset X^*$ weak* sequentially compact such that $\overline{M}^{w^*}$ has nonempty interior in the norm, then X is reflexive.*

Proof: By Theorem 3.4 the set M is weakly sequentially compact and hence, by Eberlein-Šmulyan theorem (see e.g., [7, Theorem 229]), weakly compact. Hence the weak* topology on M coincide with the weak one. Thus M is weak* compact and so weak* closed. It follows that M has nonempty interior in the norm. Hence, clearly B_{X^*} is weakly compact and thus X is reflexive. ∎

The assumption of the preceding corollary is satisfied, in particular, if B_{X^*} is weak* sequentially compact. Significant classes of such spaces are pointed out in the next corollary.

Corollary 3.6: *Let X be a weak Asplund space (or at least a Gâteaux differentiability space) such that* (I)-env$(B_X) = B_{X^{**}}$. *Then X is reflexive.*

Proof: By [3, Theorem 2.1.2] the dual unit ball B_{X^*} is weak* sequentially compact. The result then follows from Corollary 3.5. ∎

Weak Asplund spaces and Gâteaux differentiability spaces are defined via Gâteaux differentiability of convex functions. The class of weak Asplund spaces contains all weakly compactly generated spaces and, more generally, all weakly countably determined spaces and, of course, all Asplund spaces. These results and other properties of these classes can be found in [3].

Another class of Banach spaces for which Corollary 3.5 can be applied is defined using Valdivia compact spaces. Recall that a compact space K is **Valdivia** provided it is homeomorphic to a subset $K' \subset \mathbb{R}^\Gamma$ for a set Γ such that the set

$$A = \{k \in K' \colon \{\gamma \in \Gamma \colon k(\gamma) \neq 0\} \text{ is countable}\}$$

is dense in K'. More information on Valdivia compacta and their use in Banach space theory can be found in [8], the class of continuous images of Valdivia compacta is investigated in [9].

COROLLARY 3.7: *Let X be a Banach space such that* (I)-env$(B_X) = B_{X^{**}}$. *If there is an equivalent norm on X such that the respective dual unit ball is a continuous image of a Valdivia compactum (i.e., if X is a subspace of a Plichko space), then X is reflexive.*

Proof: This follows from Corollary 3.5 using the fact that any continuous image of a Valdivia compactum contains a dense sequentially compact subset. (Note that the set A from the above definition of a Valdivia compactum is sequentially compact (see e.g., [8, Lemma 1.6]) and that sequentially compact spaces are preserved by continuous mappings.) ∎

4. The unit balls of $C(K)$ spaces

The aim of this section is to prove the following example which give a negative answer to Question 1.2. The choice of ℓ_∞ is quite natural due to Theorem 3.4, as ℓ_∞ has the Grothendieck property.

Example 4.1: Let $X = \ell_\infty$. Then (I)-env$(B_X) = B_{X^{**}}$.

The space ℓ_∞ is isometric to the space of continuous functions on the compact space $\beta\mathbb{N}$. Therefore, we give a result on $C(K)$ spaces.

The following proposition characterizes those compact spaces K for which the (I)-envelope of the unit ball of $C(K)$ (this space is considered with the max-norm) is the whole bidual unit ball. Recall that by the Riesz theorem the dual to $C(K)$ can be identified with the space $M(K)$ of all finite signed Radon measures on K, the duality is given by

$$\langle \mu, f \rangle = \int_K f \,\mathrm{d}\mu,$$

the norm of a measure as an element of $C(K)^*$ coincides with its total variation. We will also work with the bidual $C(K)^{**}$. It will be sufficient to observe that the space of all bounded Borel functions equipped with the sup-norm is isometric to a subspace of $C(K)^{**}$ via the duality

$$\langle f,\mu\rangle = \int_K f\,\mathrm{d}\mu,\quad \mu\in M(K), f\colon K\to\mathbb{R} \text{ bounded Borel.}$$

Proposition 4.2: *Let K be a compact space. Then the following assertions are equivalent.*

(i) (I)-env$(B_{C(K)}) = B_{C(K)^{**}}$.

(ii) *Whenever μ_n and ν_n are two sequences of Radon probability measures on K such that μ_n and ν_m are mutually singular for each $n,m\in\mathbb{N}$ and $\varepsilon>0$ then there are $M\subset\mathbb{N}$ infinite and disjoint compact subsets A, B such that for each $n\in M$ we have $\mu_n(A)>1-\varepsilon$ and $\nu_n(B)>1-\varepsilon$.*

Proof: (i)⇒(ii) Suppose that (ii) does not hold. Let μ_n, ν_n, $n\in\mathbb{N}$, and $\varepsilon>0$ witness the failure of (ii). For each $n\in\mathbb{N}$ set $\lambda_n=\frac{1}{2}(\mu_n-\nu_n)$.

For each pair $m,n\in\mathbb{N}$ the measures μ_n and ν_m are mutually singular and hence there is a Borel set $C_{n,m}\subset K$ with $\mu_n(C_{n,m})=1$ and $\nu_m(C_{n,m})=0$. Set

$$C=\bigcup_{n=1}^{\infty}\bigcap_{m=1}^{\infty} C_{n,m}.$$

Then C is a Borel set and for each $n\in\mathbb{N}$ we have $\mu_n(C)=1$ and $\nu_n(C)=0$. The function $\chi_C-\chi_{K\setminus C}$ is a Borel function with norm 1 and satisfies

$$\langle \chi_C-\chi_{K\setminus C},\lambda_n\rangle = \lambda_n(C)-\lambda_n(K\setminus C)=1,$$

for each $n\in\mathbb{N}$.

Further, let $f\in B_{C(K)}$ be arbitrary. Let

$$A=\{x\in K\colon f(x)\ge\varepsilon/2\}\quad\text{and}\quad B=\{x\in K\colon f(x)\le-\varepsilon/2\}.$$

Then A and B are disjoint compact subsets of K and hence (by the choice of μ_n's, ν_n's and ε) there is an $n_0\in\mathbb{N}$ such that for each $n\ge n_0$ we have either $\mu_n(A)\le 1-\varepsilon$ or $\nu_n(B)\le1-\varepsilon$.

Suppose that $n\in\mathbb{N}$ is such that $\mu_n(A)\le1-\varepsilon$. Then

$$\begin{aligned}\langle\lambda_n,f\rangle=\int_K f\,\mathrm{d}\lambda_n &= \frac{1}{2}\Big(\int_K f\,\mathrm{d}\mu_n-\int_K f\,\mathrm{d}\nu_n\Big)\\ &=\frac{1}{2}\Big(\int_A f\,\mathrm{d}\mu_n+\int_{K\setminus A} f\,\mathrm{d}\mu_n-\int_K f\,\mathrm{d}\nu_n\Big)\\ &\le\frac{1}{2}(\mu_n(A)+\varepsilon/2+1)\\ &\le\frac{1}{2}(1-\varepsilon+\varepsilon/2+1)=1-\varepsilon/4\end{aligned}$$

If $\nu_n(B) \leq 1-\varepsilon$, we get by similar computation again $\langle \lambda_n, f\rangle \leq 1-\varepsilon/4$.

Hence

$$\sup_{f\in B_{C(K)}} \limsup_{n\to\infty} \langle \lambda_n, f\rangle \leq 1-\varepsilon/4,$$

and so $\chi_C - \chi_{K\setminus C}$ does not belong to the (I)-envelope of $B_{C(K)}$, by Lemma 2.3. Hence (i) is not satisfied.

(ii)⇒(i) Suppose that (ii) is satisfied. We will show that (i) is satisfied as well. First we claim that it suffices to show that $\chi_C - \chi_{K\setminus C}$ belongs to the (I)-envelope of $B_{C(K)}$ for each Borel set $C \subset K$. Indeed, if $\mu \in M(K)$ is arbitrary, then $\mu = \mu^+ - \mu^-$ and there is a Borel set C such that $\mu^-(C) = 0$ and $\mu^+(K\setminus C) = 0$. Then $\langle \chi_C - \chi_{K\setminus C}, \mu\rangle = \|\mu\|$. It follows that the functions of this form form a boundary for $B_{C(K)^{**}}$ and hence their (I)-envelope is whole bidual unit ball, by [5, Theorem 2.3]. As (I)-env((I)-env($B_{C(K)}$)) = (I)-env($B_{C(K)}$), the claim is proved.

Now suppose we have $C \subset K$ Borel. We will show, using Lemma 2.3, that $\chi_C - \chi_{K\setminus C}$ belongs to (I)-env($B_{C(K)}$). Suppose we have a sequence $\lambda_n \in B_{M(K)}$ and

$$\alpha = \inf_{n\in\mathbb{N}} \langle \chi_C - \chi_{K\setminus C}, \lambda_n\rangle > 0.$$

If $\lambda_n(C) = 0$ for infinitely many n's, then $-\lambda_n(K) = -\lambda_n(K\setminus C) \geq \alpha$ for infinitely many n's, and hence

$$\limsup_{n\to\infty} \langle \lambda_n, -1\rangle = \limsup_{n\to\infty} -\lambda_n(K) \geq \alpha.$$

Similarly, if $\lambda_n(K\setminus C) = 0$ for infinitely many n's, then

$$\limsup_{n\to\infty} \langle \lambda_n, 1\rangle = \limsup_{n\to\infty} \lambda_n(K) \geq \alpha.$$

As the constant functions 1 and -1 belong to $B_{C(K)}$, in both cases we get $\chi_C - \chi_{K\setminus C} \in$ (I)-env($B_{C(K)}$), by Lemma 2.3.

Finally, suppose that $\lambda_n(C) \neq 0$ and $\lambda_n(K\setminus C) \neq 0$ for all $n \in \mathbb{N}$ with finitely many exceptions. We can, without loss of generality, suppose that $\lambda_n(C) \neq 0$ and $\lambda_n(K\setminus C) \neq 0$ for all $n \in \mathbb{N}$. We set

$$\mu_n = \frac{|\lambda_n|_{|C}}{|\lambda_n|(C)} \quad \text{and} \quad \nu_n = \frac{|\lambda_n|_{|K\setminus C}}{|\lambda_n|(K\setminus C)},$$

for each $n \in \mathbb{N}$. Then μ_n's and ν_n's are probabilities on K and, moreover, μ_n and ν_m are mutually singular for each $m, n \in \mathbb{N}$.

Let $\varepsilon > 0$ be arbitrary. By the assumption that (ii) is satisfied there is an infinite subset $M \subset \mathbb{N}$ and disjoint closed sets A and B such that $\mu_n(A) > 1-\varepsilon$

and $\nu_n(B) > 1-\varepsilon$ for all $n \in M$. Choose a continuous function $f\colon K \to [-1,1]$ with $f_{|A} = 1$ and $f_{|B} = -1$.

Fix an arbitrary $n \in M$. Then

$$\begin{aligned}|\lambda_n|(K \setminus (A \cup B)) &\le |\lambda_n|(C \setminus A) + |\lambda_n|((K \setminus C) \setminus B)\\ &= \mu_n(C \setminus A) \cdot |\lambda_n|(C) + \nu_n((K \setminus C) \setminus B)|\lambda_n|(K \setminus C)\\ &< \varepsilon \cdot (|\lambda_n|(C) + |\lambda_n|(K \setminus C)) \le \varepsilon.\end{aligned}$$

Further,

$$\begin{aligned}\lambda_n(A) &= \lambda_n(C) - \lambda_n(C \setminus A) + \lambda_n(A \setminus C) \ge \lambda_n(C) - |\lambda_n|(C \setminus A) - |\lambda_n|(A \setminus C)\\ &= \lambda_n(C) - \mu_n(C \setminus A) \cdot |\lambda_n|(C) - \nu_n(A \setminus C) \cdot |\lambda_n|(K \setminus C)\\ &\ge \lambda_n(C) - \varepsilon \cdot (|\lambda_n|(C) + |\lambda_n|(K \setminus C)) \ge \lambda_n(C) - \varepsilon.\end{aligned}$$

Similarly,

$$\begin{aligned}\lambda_n(B) =&\lambda_n(K \setminus C) - \lambda_n((K \setminus C) \setminus B) + \lambda_n(B \setminus (K \setminus C))\\ \le&\lambda_n(K \setminus C) + \nu_n((K \setminus C) \setminus B) \cdot |\lambda_n|(K \setminus C) + \mu_n(B \cap C) \cdot |\lambda_n|(C)\\ \le&\lambda_n(K \setminus C) + \varepsilon.\end{aligned}$$

Using the three above estimates we get

$$\begin{aligned}\langle \lambda_n, f\rangle = \int_K f \mathrm{d}\lambda_n &= \lambda_n(A) - \lambda_n(B) + \int_{K\setminus(A\cup B)} f \mathrm{d}\lambda_n\\ &\ge \lambda_n(C) - \varepsilon - \lambda_n(K \setminus C) - \varepsilon - |\lambda_n|(K \setminus (A \cup B))\\ &> \lambda_n(C) - \lambda_n(K \setminus C) - 3\varepsilon \ge \alpha - 3\varepsilon.\end{aligned}$$

Hence

$$\limsup_{n\to\infty} \langle \lambda_n, f\rangle \ge \alpha - 3\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we get

$$\sup_{f\in B_{C(K)}} \limsup_{n\to\infty} \langle \lambda_n, f\rangle \ge \alpha,$$

and hence $\chi_C - \chi_{K\setminus C} \in \text{(I)-env}(B_{C(K)})$ which completes the proof. ∎

The following proposition shows that $\beta\mathbb{N}$ satisfies the condition (ii) of Proposition 4.2. To formulate it we will need the following auxiliary notions. By a finitely additive measure on $\mathbb{N}$ we mean a real-valued additive function defined on all subsets of $\mathbb{N}$. Let μ and ν be nonnegative finitely additive measures on $\mathbb{N}$. If $\varepsilon > 0$ and $A \subset \mathbb{N}$, we say that A **ε-separates μ and ν** if $\mu(A) < \varepsilon$

and $\nu(\mathbb{N} \setminus A) < \varepsilon$. It is clear that A ε-separates μ and ν if and only if $\mathbb{N} \setminus A$ ε-separates ν and μ. Further, μ and ν are called **separated** if for each $\varepsilon > 0$ there is a subset of $\mathbb{N}$ which ε-separates μ and ν.

PROPOSITION 4.3: *Let μ_n and ν_n be two sequences of finitely additive probabilities on $\mathbb{N}$. Suppose that ν_n and μ_m are separated for each $m, n \in \mathbb{N}$. Then there is an infinite set $\Lambda \subset \mathbb{N}$ such that for each $\varepsilon > 0$ there is a set $A \subset \mathbb{N}$ which ε-separates ν_n and μ_m for each $m, n \in \Lambda$.*

First we prove Example 4.1 using the previous two propositions.

Proof of Example 4.1: As the space ℓ_∞ is isometric to $C(\beta\mathbb{N})$, we may use Proposition 4.2. Let μ_n and ν_n, $n \in \mathbb{N}$ be Radon probabilities on $\beta\mathbb{N}$ such that μ_m and ν_n are mutually singular for $m, n \in \mathbb{N}$.

Let $m, n \in \mathbb{N}$ and $\varepsilon > 0$ be arbitrary. As μ_m and ν_n are mutually singular, there are disjoint closed sets $A, B \subset \beta\mathbb{N}$ such that $\mu_m(A) > 1 - \varepsilon$ and $\nu_n(B) > 1 - \varepsilon$.

Recall that $\beta\mathbb{N}$ can be viewed as the space of all ultrafilters on $\mathbb{N}$ and the basis of open sets consists of sets

$$U(M) = \{u \in \beta\mathbb{N}\colon M \in u\}, \quad M \subset \mathbb{N}.$$

For each $a \in A$ choose M_a such that $a \in U(M_a)$ and $U(M_a) \cap B = \emptyset$. Then M_a, $a \in A$, is an open cover of A and hence there are finitely many $a_1, \ldots, a_k \in A$ with $U(M_{a_1}) \cup \cdots \cup U(M_{a_k}) \supset A$. Note that

$$U(M_{a_1}) \cup \cdots \cup U(M_{a_k}) = U(M_{a_1} \cup \cdots \cup M_{a_k}).$$

Hence, there is an $M \subset \mathbb{N}$ with $A \subset U(M)$ and $B \subset \beta\mathbb{N} \setminus U(M) = U(\mathbb{N} \setminus M)$.

If λ is a Radon probability on $\beta\mathbb{N}$, then

$$\widetilde{\lambda}(M) = \lambda(U(M)), \quad M \subset \mathbb{N}$$

is a finitely additive probability on $\mathbb{N}$. Hence, by the above, $\widetilde{\mu}_n$'s and $\widetilde{\nu}_n$'s satisfy the assumptions of Proposition 4.3. Let $\Lambda \subset \mathbb{N}$ be the infinite set given by Proposition 4.3. Let $\varepsilon > 0$ be arbitrary and $M \subset \mathbb{N}$ ε-separates $\widetilde{\nu}_n$ and $\widetilde{\mu}_m$ for each $m, n \in M$. Then $\mu_n(U(M)) > 1 - \varepsilon$ and $\nu_n(U(\mathbb{N} \setminus M)) > 1 - \varepsilon$ for each $n \in \mathbb{N}$. This finishes the verification of the condition (ii) of Proposition 4.2 and hence the latter proposition concludes the proof. ∎

The rest of this section is devoted to the proof of Proposition 4.3. We will need some auxiliary results.

Lemma 4.4: *Let σ_n, $n \in \mathbb{N}$, be a sequence of nonnegative finitely additive measures on $\mathbb{N}$ such that $\sigma_n(\mathbb{N}) \le 1$ for each $n \in \mathbb{N}$. Let $A \subset \mathbb{N}$ be infinite and $\varepsilon > 0$ be such that $\sigma_n(F) < \varepsilon/2$ for each finite set $F \subset A$ and each $n \in \mathbb{N}$. Then there is an infinite subset $B \subset A$ such that $\sigma_n(B) < \varepsilon$ for each $n \in \mathbb{N}$.*

Proof: Let $\mathcal{B}$ be an uncountable family of infinite subsets of A such that each two distinct elements of $\mathcal{B}$ have finite intersection (i.e., $\mathcal{B}$ is an uncountable almost disjoint family of infinite subsets of A). The existence of such a family is a well-known fact proved, for example, in [7, Lemma 96].

We claim that for each $n \in \mathbb{N}$ there are only finitely many elements $B \in \mathcal{B}$ with $\sigma_n(B) \ge \varepsilon$. To see it let $k > 2/\varepsilon$ and $B_1, \dots, B_k$ be pairwise distinct elements of $\mathcal{B}$. Set

$$C = \bigcup\{B_i \cap B_j \colon 1 \le i < j \le k\}.$$

Then C is a finite subset of A and hence $\sigma_n(C) < \varepsilon/2$. Further, the sets $B_1 \setminus C, \dots, B_k \setminus C$ are pairwise disjoint and so there is some $i \in \{1, \dots, k\}$ with $\sigma_n(B_i \setminus C) \le 1/k < \varepsilon/2$. Then $\sigma_n(B_i) < \varepsilon$.

It follows that the family

$$\{B \in \mathcal{B} \colon \exists n \in \mathbb{N} \colon \sigma_n(B) \ge \varepsilon\}$$

is countable and hence there is $B \in \mathcal{B}$ such that $\sigma_n(B) < \varepsilon$ for all $n \in \mathbb{N}$. This completes the proof. ∎

Lemma 4.5: *Let λ_n, $n \in \mathbb{N}$, be a sequence of nonnegative finitely additive measures on $\mathbb{N}$ such that*

(a) $\lambda_n(\mathbb{N}) \le 1$ *for each* $n \in \mathbb{N}$; *and*
(b) $\lim_{n\to\infty} \lambda_n(\{k \in \mathbb{N} \colon k \ge n\}) = 0$.

Then for each $\varepsilon > 0$ there is an increasing sequence $p_0 < p_1 < p_2 < \cdots$ of positive integers such that for each $U \subset \mathbb{N}$ infinite we have

$$\liminf_{n\to\infty} \lambda_n\Bigg(\bigcup_{j\in\mathbb{N}\setminus U} \{k \in \mathbb{N} \colon p_{j-1} \le k < p_j\}\Bigg) \le \varepsilon.$$

Proof: Let $\varepsilon > 0$ be given. First we show that there is $N_0 \in \mathbb{N}$ such that for each $p > N_0$ we have

$$\liminf_{n\to\infty} \lambda_n(\{k \in \mathbb{N} \colon N_0 \le k < p\}) < \varepsilon/2.$$

Suppose that it is not the case. Then we can construct an increasing sequence of integers $1 = N_0 < N_1 < N_2 < \cdots$ such that

$$\liminf_{n\to\infty} \lambda_n(\{k \in \mathbb{N} \colon N_{j-1} \le k < N_j\}) \ge \varepsilon/2$$

for each $j \in \mathbb{N}$. Hence, for each $j \in \mathbb{N}$, we have $\lambda_n(\{k \in \mathbb{N}: N_{j-1} \le k < N_j\}) > \varepsilon/4$ for all but finitely many $n \in \mathbb{N}$. For each $j \in \mathbb{N}$ we then get that

$$\lambda_n(\{k \in \mathbb{N}: 1 \le k < N_j\}) > j\varepsilon/4$$

for all but finitely many $n \in \mathbb{N}$. By choosing $j > 4/\varepsilon$ we get a contradiction with the assumption that $\lambda_n(\mathbb{N}) \le 1$ for each $n \in \mathbb{N}$.

Further we choose $N > N_0$ such that $\lambda_n(\{k \in \mathbb{N}: k \ge n\}) < \varepsilon/2$ for each $n > N$. By the previous paragraph we can construct an increasing sequence of integers $N = p_0 < p_1 < p_2 < \cdots$ such that $\lambda_{p_n}(\{k \in \mathbb{N}: N \le k < p_{n-1}\}) < \varepsilon/2$ for each $n \in \mathbb{N}$. It remains to show that this sequence satisfies the required condition.

Let $U \subset \mathbb{N}$ be infinite. Set

$$M = \bigcup_{j \in \mathbb{N} \setminus U} \{k \in \mathbb{N}: p_{j-1} \le k < p_j\}.$$

Then, for each $n \in U$, we have

$$\begin{aligned}\lambda_{p_n}(M) &\le \lambda_{p_n}(\{k \in \mathbb{N}: N \le k < p_{n-1}\}) + \lambda_{p_n}(\{k \in \mathbb{N}: k \ge p_n\}) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

The assertion then easily follows. ∎

Lemma 4.6: *Let ν_n and μ_n, $n \in \mathbb{N}$, be finitely additive probabilities on $\mathbb{N}$ such that for each $n \in \mathbb{N}$ and each $\varepsilon > 0$ there is a set $A \subset \mathbb{N}$ which ε-separates μ_m and ν_n for each $m \in \mathbb{N}$. Then there is an infinite set $\Lambda \subset \mathbb{N}$ such that for each $\varepsilon > 0$ there is $A \subset \mathbb{N}$ which ε-separates μ_m and ν_n for each $m \in \mathbb{N}$ and $n \in \Lambda$.*

Proof: Let $\varepsilon > 0$. We first show that there is $\Lambda_0 \subset \mathbb{N}$ infinite and $A \subset \mathbb{N}$ which ε-separates μ_m and ν_n for each $m \in \mathbb{N}$ and $n \in \Lambda_0$.

For each $n \in \mathbb{N}$ there is a set $B_n \subset \mathbb{N}$ which $\varepsilon/2^{n+2}$ separates μ_m and ν_n for each $m \in \mathbb{N}$. Set $C_n = B_1 \cup \cdots \cup B_n$ for $n \in \mathbb{N}$. Then

$$\begin{aligned}\mu_m(C_n) &\le \mu_m(B_1) + \cdots + \mu_m(B_n) < \varepsilon/2^3 + \cdots + \varepsilon/2^{n+2} < \varepsilon/4 \\ \nu_n(\mathbb{N} \setminus C_n) &\le \nu_n(\mathbb{N} \setminus B_n) < \varepsilon/2^{n+2}.\end{aligned}$$

for all $m, n \in \mathbb{N}$. We define finitely additive measures λ_n, $n \in \mathbb{N}$, on $\mathbb{N}$ by the formula

$$\lambda_n(A) = \nu_n\Big(\bigcup_{k \in A} C_{k+1} \setminus C_k\Big), \quad A \subset \mathbb{N}, n \in \mathbb{N}.$$

It is clear that each λ_n is a nonnegative finitely additive measure on $\mathbb{N}$ satisfying $\lambda_n(\mathbb{N}) \le 1$. Moreover,

$$\lambda_n(\{k \in \mathbb{N}\colon k \ge n\}) = \nu_n\left(\bigcup_{k=1}^{\infty} C_k \setminus C_n\right) \le \nu_n(\mathbb{N} \setminus C_n) < \frac{\varepsilon}{2^{n+2}},$$

and hence the sequence λ_n satisfies the assumptions of Lemma 4.5. Let $p_0 < p_1 < p_2 < \cdots$ be the sequence corresponding to λ_n's and $\varepsilon/2$ by Lemma 4.5.

Further, we define finitely additive measures σ_n, $n \in \mathbb{N}$, on $\mathbb{N}$ by the formula

$$\sigma_n(U) = \mu_n\left(\bigcup_{j \in U} C_{p_j} \setminus C_{p_{j-1}}\right), \quad U \subset \mathbb{N}, n \in \mathbb{N}.$$

Then clearly $\sigma_n(\mathbb{N}) \le 1$ for each $n \in \mathbb{N}$ and $\sigma_n(F) < \varepsilon/4$ for each finite $F \subset \mathbb{N}$. It follows from Lemma 4.4 that there is $U \subset \mathbb{N}$ infinite such that $\sigma_n(U) < \varepsilon/2$ for all $n \in \mathbb{N}$. Set

$$A = C_{p_0} \cup \bigcup_{j \in U} C_{p_j} \setminus C_{p_{j-1}}.$$

Then we have for all $n \in \mathbb{N}$

$$\mu_n(A) = \mu_n(C_{p_0}) + \sigma_n(U) < \varepsilon/4 + \varepsilon/2 < \varepsilon.$$

Further, for each $n \in \mathbb{N}$ we have

$$\begin{aligned}\nu_n(\mathbb{N} \setminus A) &\le \nu_n(\mathbb{N} \setminus C_n) + \nu_n\left(\bigcup_{j \in \mathbb{N}\setminus U} C_{p_j} \setminus C_{p_{j-1}}\right)\\ &< \varepsilon/4 + \lambda_n\left(\bigcup_{j \in \mathbb{N}\setminus U} \{k \in \mathbb{N}\colon p_{j-1} \le k < p_j\}\right),\end{aligned}$$

thus

$$\liminf_{n\to\infty} \nu_n(\mathbb{N} \setminus A) \le \varepsilon/4 + \varepsilon/2,$$

hence $\nu_n(\mathbb{N}\setminus A) < \varepsilon$ for infinitely many $n \in \mathbb{N}$. It follows that there are infinitely many $n \in \mathbb{N}$ such that A ε-separates μ_m and ν_n for all $m \in \mathbb{N}$.

To finish the proof we construct by induction infinite subsets $\Lambda_1 \supset \Lambda_2 \supset \cdots$ of $\mathbb{N}$ such that for each $k \in \mathbb{N}$ there is a set $A_k \subset \mathbb{N}$ which $\frac{1}{2^{k+1}}$-separates μ_m and ν_n for each $m \in \mathbb{N}$ and $n \in \Lambda_k$. We can choose, by induction, pairwise distinct elements $a_k \in \Lambda_k$, $k \in \mathbb{N}$, and set $\Lambda = \{a_k\colon k \in \mathbb{N}\}$. Then Λ is an infinite subset of $\mathbb{N}$. We will show that it satisfies the required condition.

Let $\varepsilon > 0$. Choose $k \in \mathbb{N}$ such that $\frac{1}{2^k} < \varepsilon$. For each $j \in \{1, \dots, k-1\}$ choose $D_j \subset \mathbb{N}$ which $\frac{1}{k2^{k+1}}$-separates μ_m and ν_{a_j} for each $m \in \mathbb{N}$. Set $A = D_1 \cup \dots \cup D_{k-1} \cup A_k$. Then we have for each $m \in \mathbb{N}$

$$\mu_m(A) \leq (k-1) \cdot 1/(k2^{k+1}) + 1/2^{k+1} < 1/2^k < \varepsilon.$$

If $j \in \{1, \dots, k-1\}$, then

$$\nu_{a_j}(\mathbb{N} \setminus A) \leq \nu_{a_j}(\mathbb{N} \setminus C_j) < 1/(k2^{k+1}) < \varepsilon.$$

Finally, if $j \geq k$, then $a_j \in \Lambda_k$ and hence

$$\nu_{a_j}(\mathbb{N} \setminus A) \leq \mu_{a_j}(\mathbb{N} \setminus A_k) < 1/2^{k+1} < \varepsilon.$$

This completes the proof. ∎

Proof of Proposition 4.3: Let μ_n, ν_n, $n \in \mathbb{N}$, satisfy the assumptions of Proposition 4.3.

Let $k \in \mathbb{N}$ be fixed. We apply Lemma 4.6 to the constant sequence $\{\nu_k\}_{m \in \mathbb{N}}$ in place of $\{\mu_m\}_{m \in \mathbb{N}}$ and $\{\mu_n\}_{n \in \mathbb{N}}$ in place of $\{\nu_n\}_{n \in \mathbb{N}}$. We obtain an infinite set $\Lambda_0 \subset \mathbb{N}$ such that for each $\varepsilon > 0$ there is an $A \subset \mathbb{N}$ which ε-separates ν_k and μ_n for each $n \in \Lambda_0$.

By induction we can construct infinite subsets $\Lambda_1 \supset \Lambda_2 \supset \Lambda_3 \supset \cdots$ of $\mathbb{N}$ such that whenever $n \in \mathbb{N}$ and $\varepsilon > 0$ then there is a set $A \subset \mathbb{N}$ which ε-separates ν_n and μ_m for each $m \in \Lambda_n$. Choose pairwise distinct elements $c_n \in \Lambda_n$, $n \in \mathbb{N}$, and set $\Lambda' = \{c_n : n \in \mathbb{N}\}$. Then Λ' is infinite. Moreover, for all $n \in \mathbb{N}$ and $\varepsilon > 0$ there is a set $A \subset \mathbb{N}$ which ε-separates ν_n and μ_m for all $m \in \Lambda_0$.

To see this, fix $n \in \mathbb{N}$ and $\varepsilon > 0$. Choose $B \subset \mathbb{N}$ which $\frac{\varepsilon}{2}$-separates ν_n and μ_m for all $m \in \Lambda_n$. Further, for each $j = 1, \dots, n-1$ choose $C_j \subset \mathbb{N}$ which $\frac{\varepsilon}{2n}$-separates ν_n and μ_{c_j}. Then it is easy to check that the set $A = B \cup C_1 \cup \dots \cup C_{n-1}$ ε-separates ν_n and μ_m for each $m \in \Lambda'$.

We can, without loss of generality, assume that $\Lambda' = \mathbb{N}$, which can be rephrased as:

For each $n \in \mathbb{N}$ and each $\varepsilon > 0$ there is a set $A \subset \mathbb{N}$ which ε-separates μ_m and ν_n for each $m \in \mathbb{N}$.

In other words, the assumption of Lemma 4.6 is satisfied. The infinite set Λ provided by Lemma 4.6 is the one required by Proposition 4.3. This completes the proof. ∎

5. The case of complex spaces

The aim of this section is to show that the results formulated and proved in the previous sections for real spaces can be easily transferred to complex spaces.

If X is a complex Banach space we denote by X_R the space X considered as a real space. Then we have:

(a) The identity map X onto X_R is a real-linear isometry and weak-to-weak homeomorphism.

(b) The map $\phi\colon X^* \to X_R^*$ defined by

$$\phi(\xi)(x) = \operatorname{Re} \xi(x), \quad x \in X,\, \xi \in X^*,$$

is a real-linear isometry, weak-to-weak and weak*-to-weak* homeomorphism of X^* onto X_R^*.

(c) The map $\psi\colon X^{**} \to X_R^{**}$ defined by

$$\psi(F)(\eta) = \operatorname{Re} F(\phi^{-1}(\eta)), \quad \eta \in X_R^*,\, F \in X^{**},$$

is a real-linear isometry and weak*-to-weak* homeomorphism.

(d) Let $\kappa\colon X \to X^{**}$ and $\kappa_R\colon X_R \to X_R^{**}$ denote the canonical embeddings. Then $\kappa_R = \psi \circ \kappa$.

The fact that the identity of X onto X_R is a real-linear isometry is obvious. The map ϕ is obviously real linear. The fact that it is an onto isometry is easy and it is a standard step in proving a complex version of Hahn–Banach theorem (see e.g., [7, pp. 28–29]). Then it follows easily that the identity of X onto X_R is a weak-to-weak homeomorphism and that ϕ is a weak*-to-weak* homeomorphism.

It follows that ϕ is a linear isometry of $(X^*)_R$ onto X_R^*, and hence ϕ is a weak-to-weak homeomorphism.

If we apply the previous two paragraphs to X^* instead of X we get easily (c). The assertion (d) is obvious.

Let us now suppose that X is a complex Banach space and $B \subset X$. From the above comparison of X and X_R it follows that the (I)-envelope of B in X_R^{**} is the image by ψ of the (I)-envelope in X^{**}. Further, the dual unit ball of B_{X^*} is weak* angelic if and only if $B_{X_R^*}$ is angelic and X has the Grothendieck property if and only if X_R has the Grothendieck property. It follows that Theorems 3.1 and 3.4 are valid also for complex spaces. The same is true for the corollaries to Theorem 3.4.

As for Example 3.2, its proof works for the complex space of complex-valued continuous functions $C([0,\omega_1],\mathbb{C})$ as well.

The only not completely trivial fact is that Example 4.1 can be transferred to complex spaces, namely, that its statement is valid for complex version of ℓ_∞.

We note first that Lemma 2.1 is true for complex spaces if we replace $\xi(x_n)$ and $\eta(x_n)$ by $\operatorname{Re}\xi(x_n)$ and $\operatorname{Re}\eta(x_n)$ in assertions (2)–(4). Analogous adaptation could be done with Lemma 2.3. Further, note that for space of complex-valued continuous functions the Riesz theorem is true as well (the dual to $C(K,\mathbb{C})$ is the space of all complex-valued Radon measures on K with the total variation norm) and bounded Borel functions from K to $\mathbb{C}$ can be viewed as a subspace of $C(K,\mathbb{C})^{**}$ by the same duality as that described in the beginning of Section 4 for space of real functions.

The fact that if X is the complex version of ℓ_∞, then (I)-env(B_X) = (I)-env$(B_{X^{**}})$ follows immediately from the following proposition.

Proposition 5.1: *Let K be a compact space. Then the following assertions are equivalent.*

(1) (I)-env$(B_{C(K,\mathbb{R})}) = B_{C(K,\mathbb{R})^{**}}$
(2) (I)-env$(B_{C(K,\mathbb{C})}) = B_{C(K,\mathbb{C})^{**}}$.

Proof: 2⇒1 Suppose that (1) is not true. Then the condition (ii) of Proposition 4.2 is violated. Let μ_n, ν_n, $n\in\mathbb{N}$, and $\varepsilon>0$ witness that. By the proof of the implication (i)⇒(ii) of Proposition 4.2 there is a Borel set $C\subset K$ such that $\mu_n(C)=1$ and $\nu_n(C)=0$ for each $n\in\mathbb{N}$. If we set $\lambda_n=\frac{1}{2}(\mu_n-\nu_n)$, we get

$$\langle \chi_C-\chi_{K\setminus C},\lambda_n\rangle = 1$$

for all $n\in\mathbb{N}$ and

$$\sup_{f\in B_{C(K,\mathbb{R})}}\ \limsup_{n\to\infty}\langle\lambda_n,f\rangle\le 1-\varepsilon/4.$$

As $\|\operatorname{Re} f\|\le\|f\|$ for each $f\in C(K,\mathbb{C})$, we have

$$\begin{aligned}\sup_{f\in B_{C(K,\mathbb{C})}}\ \limsup_{n\to\infty}\operatorname{Re}\langle\lambda_n,f\rangle &= \sup_{f\in B_{C(K,\mathbb{C})}}\ \limsup_{n\to\infty}\langle\lambda_n,\operatorname{Re} f\rangle\\ &\le \sup_{f\in B_{C(K,\mathbb{R})}}\ \limsup_{n\to\infty}\langle\lambda_n,f\rangle\\ &\le 1-\varepsilon/4.\end{aligned}$$

Hence by the complex version of Lemma 2.3,

$$\chi_C-\chi_{K\setminus C}\notin \text{(I)-env}(B_{C(K,\mathbb{C})}).$$

1⇒2 Suppose that (1) is satisfied. Then the assertion (ii) of Proposition 4.2 is satisfied as well. The proof will proceed in several steps.

STEP 1: *For any $\mu \in C(K,\mathbb{C})^*$ there is a Borel function $f\colon K \to \mathbb{C}$ such that $|f(x)| = 1$ for all $x \in K$ and $\int_K f \mathrm{d}\mu = \|\mu\|$.*

As μ is absolutely continuous with respect to $|\mu|$, there is, by a consequence to the Radon-Nikodým theorem (see e.g., [12, Theorem 6.12]) a $|\mu|$-measurable function $h\colon K \to \mathbb{C}$ such that $|h(x)| = 1$ $|\mu|$-almost everywhere and

$$\int_K g \mathrm{d}\mu = \int_K gh \mathrm{d}|\mu|$$

for each integrable function g. We can choose a Borel function $f\colon K \to \mathbb{C}$ such that $|f(x)| = 1$ for all $x \in K$ and $f(x) = \overline{h(x)}$ $|\mu|$-almost everywhere. Then

$$\int_K f \mathrm{d}\mu = \int_K fh \mathrm{d}|\mu| = \int_K \overline{h}h \mathrm{d}|\mu| = |\mu|(K) = \|\mu\|,$$

and thus f is the sought function.

STEP 2: *It suffices to show that $f \in$ (I)-env$(B_{C(K,\mathbb{C})})$ for each Borel function $f\colon K \to \mathbb{C}$ with $|f(x)| = 1$ for each $x \in K$.*

By Step 1 these functions form a boundary of $B_{C(K,\mathbb{C})^{**}}$ (we use the identification of $C(K,\mathbb{C})^{**}$ with $C(K,\mathbb{C})^{**}_R$ from the beginning of this section). Hence the assertion follows from [5, Theorem 2.3] and the fact that (I)-env((I)-env$(B_{C(K,\mathbb{C})})) =$ (I)-env$(B_{C(K,\mathbb{C})})$.

STEP 3: *Let $k \in \mathbb{N}$ and μ_n^j, $j = 1, \dots, k$, $n \in \mathbb{N}$, be Radon probabilities on K such that μ_m^j and μ_n^l are mutually singular whenever j and l are distinct numbers from $\{1, \dots, k\}$ and $m, n \in \mathbb{N}$ are arbitrary. Then for each $\varepsilon > 0$ there are an infinite subset $M \subset \mathbb{N}$ and pairwise disjoint compact sets $A_1, \dots, A_k$ such that $\mu_n^j(A_j) > 1 - \varepsilon$ for each $j \in \{1, \dots, k\}$ and $n \in M$.*

Suppose $\varepsilon > 0$. Let $(j_1, l_1), \dots, (j_N, l_N)$ be an enumeration of all pairs $(j, l) \in \{1, \dots, k\}^2$ with $j < l$. As the assertion (ii) of Proposition 4.2 is satisfied, we may construct infinite sets

$$\mathbb{N} \supset M_1 \supset M_2 \supset \cdots \supset M_N$$

and pairs (J_i, L_i) of disjoint compact sets such that

$$\mu_n^{j_i}(J_i) > 1 - \varepsilon/k \quad \text{and} \quad \mu_n^{l_i}(L_i) > 1 - \varepsilon/k \quad \text{for } n \in M_i,\ i = 1, \dots, N.$$

Set $M = M_N$ and

$$A_j = \bigcap_{j=j_i} J_i \cap \bigcap_{j=l_i} L_i, \quad j = 1, \dots, k.$$

Then M and $A_1, \dots, A_k$ clearly have the required properties.

Step 4: *$f \in$ (I)-env$(B_{C(K,\mathbb{C})})$ for each Borel function $f\colon K \to \mathbb{C}$ with $|f(x)| = 1$ for each $x \in K$.*

We will use the complex version of Lemma 2.3. Suppose we have a sequence (of complex measures) $\lambda_n \in B_{M(K)}$ and

$$\alpha = \inf_{n\in\mathbb{N}} \mathrm{Re}\langle f, \lambda_n\rangle > 0.$$

Let $\varepsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such that $|e^{2\pi i/N} - 1| < \varepsilon$ and set

$$B_j = f^{-1}\Big(\Big\{e^{it}\colon 2(j-1)\pi/N \le t < 2j\pi/N\Big\}\Big), \quad j = 1, \dots, N.$$

Then $B_1, \dots, B_N$ is a partition of K into Borel sets. There is an infinite set $\Lambda \subset \mathbb{N}$ and a nonempty set $F \subset \{1, \dots, N\}$ such that

$$F = \{j \in \{1, \dots, N\}\colon |\lambda_n|(B_j) > 0\}, \quad \text{for each } n \in \Lambda.$$

Renumerate λ_n, $n \in \Lambda$, as σ_n, $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ and $j \in F$ set

$$\mu_n^j = \frac{|\sigma_n|_{|B_j}}{|\sigma_n|(B_j)}.$$

Then each μ_n^j is a probability and, moreover, μ_n^j and μ_m^l are mutually singular whenever j and l are distinct and $m, n \in \mathbb{N}$ arbitrary. Hence, by Step 3, there are $M \subset \mathbb{N}$ infinite and pairwise disjoint compact sets A_j, $j \in F$, such that $\mu_n^j(A_j) > 1 - \varepsilon/N$ for each $j \in F$ and $n \in M$. Let $g\colon K \to \mathbb{C}$ be a continuous function such that $|g(x)| \le 1$ for each $x \in K$ and $g(x) = e^{2(j-1)\pi i/N}$ for $x \in A_j$, $j \in F$.

We have for each $n \in M$

$$|\sigma_n|\Big(K \setminus \bigcup_{j\in F} A_j\Big) \le \sum_{j\in F} \mu_n^j(B_j \setminus A_j) \cdot |\sigma_n|(B_j) < \varepsilon/N \le \varepsilon.$$

Further,

$$|\sigma_n|(A_j \setminus B_j)| = \sum_{l\in F\setminus\{j\}} \mu_n^l(A_j \cap B_l) \cdot |\sigma_n|(B_l) < \varepsilon/N$$

for each $j \in F$. Hence

$$\begin{aligned}\Big|\int_K g \mathrm{d}\sigma_n - \int_K f \mathrm{d}\sigma_n\Big| \\ \le \int_K |g - f| \mathrm{d}|\sigma_n|\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j\in F} \int_{A_j\cap B_j} |f-g| \mathrm{d}|\sigma_n| + \sum_{j\in F} \int_{A_j\setminus B_j} |f-g| \mathrm{d}|\sigma_n| \\
&\quad + \int_{K\setminus \bigcup_{j\in F} A_j} |f-g| \mathrm{d}|\sigma_n| \\
&\leq \sum_{j\in F} |e^{2j\pi i/N} - e^{2(j-1)\pi i/N}| \cdot |\sigma_n|(A_j) + 2\sum_{j\in F}\frac{\varepsilon}{N} + 2|\sigma_n|\Big(K\setminus \bigcup_{j\in F} A_j\Big) \\
&< |e^{2\pi i/N}-1| \cdot \sum_{j\in F} |\sigma_n|(A_j) + 4\varepsilon < 5\varepsilon.
\end{aligned}
$$

It follows that

$$\operatorname{Re}\langle \sigma_n, g\rangle \geq \operatorname{Re}\langle f, \sigma_n\rangle - 5\varepsilon \geq \alpha - 5\varepsilon$$

for all $n \in M$. Hence

$$\limsup_{n\to\infty} \operatorname{Re}\langle \lambda_n, g\rangle \geq \limsup_{n\to\infty} \operatorname{Re}\langle \sigma_n, g\rangle \geq \alpha - 5\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we get

$$\sup_{g\in B_{C(K,\mathbb{C})}} \limsup_{n\to\infty} \operatorname{Re}\langle \lambda_n, g\rangle \geq \alpha,$$

which completes the proof. ∎

6. Final remarks and open questions

In this section we collect some natural questions on (I)-envelopes of bounded closed convex sets.

QUESTION 6.1: *Let X be a Banach space and $B \subset X$ be a closed convex set such that* $\operatorname{span} B$ *is dense in X. Is the (I)-envelope of B equal to the set of all weak* sequentially continuous functionals from $\overline{B}^{w^*}$?*

Proposition 2.4 says that, even without assuming that $\operatorname{span} B$ is dense, every element of (I)-env(B) is weak* sequentially continuous and that (I)-env(B) are exactly weak* sequentially elements of $\overline{B}^{w^*}$ provided B_{X^*} is weak* sequentially compact. Further, Example 2.5 shows that the answer to the previous question would be negative if we did not require $\operatorname{span} B$ to be dense.

Hence our question seems to be quite natural. It is open, in particular, if $B = B_X$. A special case of the previous question is the following one.

Question 6.2: *Let X be a Banach space with the Grothendieck property. Is then* (I)-env$(B_X) = B_{X^{**}}$*?*

Theorem 3.4 asserts that the converse is true. Example 4.1 shows that for $X = \ell_\infty$ the answer is positive.

A further related question is the following one.

Question 6.3: *Let X be a Banach space such that* (I)-env$(B_X) = B_{X^{**}}$. *Let B be the unit ball of an equivalent norm on X. Is then* (I)-env$(B) = \overline{B}^{w^*}$*?*

The answer to this question is positive if Question 6.2 has positive answer. Note, that is is easy to check that (I)-env(B) is the unit ball of an equivalent norm X^{**} but it is not clear whether it is equal to $\overline{B}^{w^*}$.

Question 6.4: *Let X be a Banach space with B_{X^*} weak* sequentially compact and $B \subset X$ be a bounded closed convex set with* (I)-env$(B) = \overline{B}^{w^*}$. *Is then B weakly compact?*

Theorem 3.1 says that the answer is positive if B_{X^*} is weak* angelic. Further, by Corollary 3.5 the answer is positive if $B = B_X$.

The answer is further positive if $X = C[0, \omega_1]$. Indeed, in this case B_{X^*} is weak* sequentially compact and hence the (I)-envelope of any B is the set of all weak* sequentially continuous elements of $\overline{B}^{w^*}$. Further, weak* sequentially continuous functionals from $X^{**} = \ell_\infty[0, \omega_1]$ are exactly those elements of $\ell_\infty[0, \omega_1]$ which are continuous at all points with the possible exception of the point ω_1. If B is not weakly compact, then it is not weakly countably compact and hence there is a countable set $C \subset B$ which has no weak accumulation point in B. Let c be a weak* accumulation point of C in $\overline{B}^{w^*}$. Then c is continuous at ω_1 and is not continuous on $[0, \omega_1]$. It follows that c is not weak* sequentially continuous and hence $c \notin$ (I)-env(B).

It is not clear whether an analogous argument could work in general.

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Added in proof. After this paper was finished and accepted for publication, the author found answers to the mentioned open problems. Namely, the answers to the first three questions are negative. In fact, the only Banach spaces for which the equality (I)-env$(B_X) = B_{X^{**}}$ holds in any renorming are the reflexive ones. This result is contained in a paper *(I)-envelopes of unit balls and James'* *characterization of reflexivity* by the author (Studia Math. **182** (2007), no. 1, 29–40). Moreover, the answer to Question 6.4 is easily seen to be positive.

Further, Theorem 3.4 was independently proved by O. Nygaard (see Ann. Math. Inf. **32** (2005), 125–127).

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